k-core percolation in four dimensions

Giorgio Parisi[1] and Tommaso Rizzo[2]
[1]*Dipartimento di Fisica, Università di Roma "La Sapienza," P. le Aldo Moro 2, 00185 Roma, Italy*
[2]*"E. Fermi" Center, Via Panisperna 89 A, Compendio Viminale, 00184 Roma, Italy*
(Received 21 March 2007; published 27 August 2008)

The k-core percolation on the Bethe lattice has been proposed as a simple model of the jamming transition because of its hybrid first-order–second-order nature. We investigate numerically k-core percolation on the four-dimensional regular lattice. For $k=4$, the presence of a discontinuous transition is clearly established but its nature is strictly first-order. In particular, the k-core density displays no singular behavior before the jump and its correlation length remains finite. For $k=3$, the transition is continuous.

DOI: 10.1103/PhysRevE.78.022101 PACS number(s): 64.60.A−

Since its introduction [1], k-core percolation has been proven to be relevant in a variety of contexts ranging from glassy systems, to magnetic systems, to computer memory storage [2,3]. The problem, also referred to as "bootstrap percolation," is defined as follows. The sites of a given lattice are populated with probability p. Then each site with fewer than k neighbors is removed, the procedure being iterated until each site has at least k neighbors. The remaining occupied sites are referred to as the k-core.

On regular lattices in dimension d, the model exhibits two different behaviors depending on the value of k. If $k>d$, the cluster must be extended in order to survive the culling process but it is completely decimated for any $p<1$ in the large size limit [4]. The behavior for large but finite system size has also been investigated [5–8], but strong disagreement between the theoretical predictions and numerical simulations has been found. The highly nontrivial origin of this discrepancy was clarified only recently [9]. If $k\leqslant d$, it is easy to realize that there are small "self-sustained" structures (e.g., d-dimensional hypercubes) that can survive culling irrespective of their environment. In this case, the k-core always exists and the problem is rather whether it percolates or not and what is the nature of the percolation transition [10].

On the Bethe lattice, the k-core percolation transition is known to be discontinuous [11]. Starting from high values of p, the density of the k-core drops discontinuously to zero at p_c. The transition, however, is not simply first order; the density near the transition is given by $\rho(p)\simeq\rho(p_c)+b(p-p_c)^{1/2}$. Furthermore, it was recently pointed out that the transition is also accompanied by diverging correlation lengths. This behavior has motivated the proposal of k-core percolation as a model of the jamming transition [12,13]. There is indeed evidence that this transition has a mixed character [14]. The hybrid character of the transition was also found for k-core percolation on complex networks [15].

This has brought new interest into the question of whether the hybrid nature of the transition in the Bethe lattice survives on other lattices in finite dimensions. This is certainly not the case for cubic lattices in $d\leqslant 3$. Indeed, for $k=2$ the transition is continuous and has the same critical point of ordinary percolation [16]. For $k=3$ and $d=3$, the transition is continuous [11,17–19] with exponents consistent with those of ordinary $d=3$ percolation [18]. These results are valid on cubic lattices and they do not exclude the possibility of a mixed transition for $d\leqslant 3$ provided the structure of the lattice or the constraints are different. Indeed, recently a two-dimensional model with a mixed transition was exhibited [20] and numerical evidence of a mixed transition in another two-dimensional model was reported in [12]. As for regular lattices, an expansion in powers of $1/d$ has proven that turning on dimension perturbatively does not destroy the mixed nature of the transition [10], thus suggesting that the hybrid transition may exist for some $(d>3,2<k<d+1)$. In this work, we investigate numerically k-core percolation on the four-dimensional hypercubic lattice. For $k=3$, we have found a continuous transition and we did not further investigate the critical behavior. In the case ($d=4$, $k=4$), we find negative results concerning the hybrid transition: while the presence of a discontinuous transition is clearly established, it is strictly first order. More precisely for $k=4$, at a critical value $p_c=0.6885(5)$ the system has a phase transition from a high p phase where there is a giant cluster with a finite density to a low-p phase where there is not. The density of the giant cluster is given by the k-core density minus the density ρ_{small} of the small clusters, where $\rho_{\mathrm{small}}\approx 0.04$ near the transition, therefore the critical properties of the giant cluster can be safely extracted from the total density in the percolating phase. The k-core density exhibits a discontinuous transition, jumping from $\rho_c^+=0.567(4)$ to $\rho_c^-=0.044(2)$. However, the density displays no singular behavior at the transition, and the correlation length extracted from k-core correlation function $G(i,j)=\langle\nu_i\nu_j\rangle-\langle\nu_i\rangle\langle\nu_j\rangle$ (where ν_1 is 1 on the k-core and 0 otherwise [10]) remains finite, $\xi_c<10$.

We started the numerical investigations considering hypercubic lattices with periodic boundary conditions (PBCs). As we will show, behavior with PBCs will enable us to access a metastable regime that is quite different from the bulk behavior.

We note that the data presented in this paper are always for a single run at different sizes, i.e., a given realization of the system. Indeed, due to time and memory constraints, few lattices of the largest size ($L=320$) could be studied; however, we checked that the sample-to-sample fluctuations are practically irrelevant to estimate the k-core density as soon as we consider system sizes larger than $L=32$. For instance, the inset of Fig. 4 clearly shows that the bulk density estimates computed from hypercubes of different size $L=100$, $150,200$ inside one sample do not show significant devia-

 ©2008 The American Physical Society

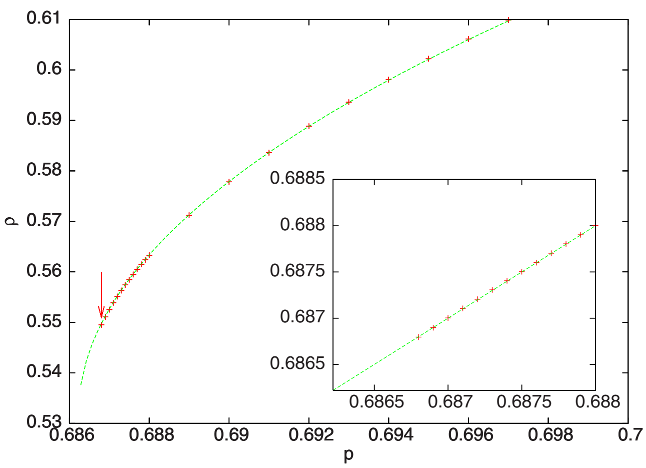

FIG. 1. (Color online) k-core density vs probability with $k=4$ for a four-dimensional sample of size $L=320$ and periodic boundary conditions. The arrow marks a discontinuous transition in the density that jumps to $\rho \approx 0.05$ at lower values of p. The data are fit in the region near the transition with the function $\rho(p) \simeq \rho(\hat{p}_c) + b(p - \hat{p}_c)^{1/2}$ but the actual transition probability $\hat{p}$ (marked by the arrow) is above the probability $\hat{p}_c$ estimated from the fit. Inset: plot of $\hat{p}_c + [\rho(p) - \rho(\hat{p}_c)]^2/b^2$ near the transition as a function of p; the fitting curve maps onto the line $y=x$.

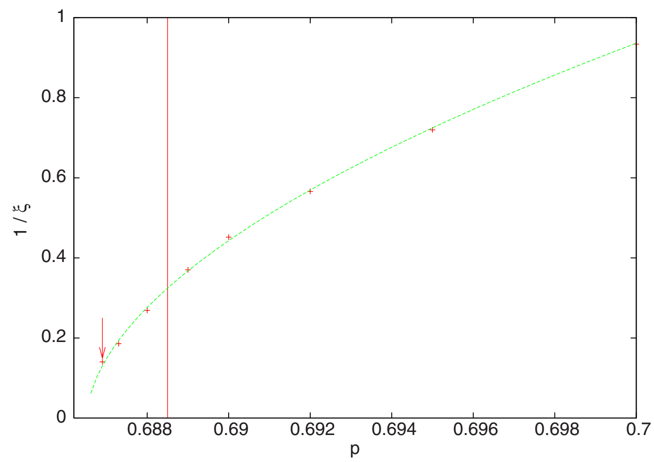

FIG. 2. (Color online) Inverse of the correlation length ξ vs probability for the same sample as Fig. 1. The arrow marks the point of the actual transition $\hat{p}=0.6869$ on this finite-size realization. Although the behavior is consistent with a divergence at $\hat{p}_c \approx 0.6862$, in all samples studied we never observed a correlation length at the actual transition exceeding $\xi \approx 8-10$, i.e., large but much smaller than the sample size $L=320$. The vertical line marks the value of the true critical probability $p_c=0.6885(5)$; see text.

tions. A good self-averaging behavior is also suggested by the smoothness of the density curve on a single sample.

In Fig. 1, we plot the density of the k-core for a sample of size $L=320$ with PBCs, corresponding to $O(10^{10})$ sites. The density of the k-core has a discontinuous transition at $\hat{p} = 0.6869$, where it jumps from a high-density percolating phase to a low-density non-percolating phase $\rho \approx 0.05$. The behavior appears to be consistent with a singular behavior at the transition, but a careful study of the data in order to extract the critical p and the exponent β shows some inconsistencies. Indeed, the curve seems to be fit at best with the exponent $\beta = \frac{1}{2}$ (the mean-field Bethe-lattice value) but with a value of the critical probability $\hat{p}_c \approx 0.6862$ definitively *lower* than that at which the transition is actually observed, i.e., $\hat{p}=0.6869$. In order to assess whether this behavior can be considered a finite-size effect, we investigated the k-core spatial correlation function.

In Fig. 2, we plot the inverse correlation length ξ as a function of p for the same sample; again the plot is apparently consistent with a divergence of ξ at a value of p slightly lower than the value at which the transition is actually observed (marked with an arrow in Fig. 2). In principle, this could be a finite-size effect but the problem is that the value of ξ at the actual transition is large ($\xi \approx 8-10$) but *not comparable* with the size of the system ($L=320$). Actually, we were not able to observe a correlation length bigger than ten lattice spacings *independently of the size of the system* (*up to* $L=320$), and it seems highly unlikely that this divergence drives the transition. These features made us suspect that the actual mechanism driving the transition is not the divergence of the correlation length but rather the nucleation of droplets of the low-density phase. Due to the periodic boundary conditions, these nucleation centers are originally absent in the system but they appear as spatial fluctuations of the density when the correlation length is sufficiently big leading to the transition. We tested this idea by putting some nucleation centers (i.e., empty hypercubes of size l) by hand in the sample, and we checked that this procedure shifts the transition at higher values of p, although the size of the hypercubes ($l=20,40$) is small with respect to the size of the system and large with respect to the correlation length. According to this interpretation, the percolating phase is unstable and can be observed only because nucleation centers in finite-size systems with periodic boundary conditions are extremely rare.

In order to assess the validity of this interpretation and to determine whether there is a true percolation transition at higher values of p, we considered systems with completely empty boundaries. These boundary conditions guarantee that the system is completely isolated from the outside. As a consequence, the density at finite size is a *lower bound* to the density in the thermodynamic limit. Furthermore, it turned out that in this case numerical methods are rather safe for extrapolating the behavior of the infinite system at variance with the more delicate $k > d$ case, where finite-size effects are extremely large as mentioned above [9]. The possibility of choosing these boundary conditions is a special feature of the case $k \leq d$ because otherwise no site can resist culling if the boundaries of the hypercube are empty.

In order to estimate the density at a given value of p, we generated lattices of increasing size. In Fig. 3, we plot the results for the *total density* ρ_L at various sizes L. The total density ρ_L is affected by the presence of the empty boundaries and *is an increasing function of* L. The observed monotonicity property allows us to safely conclude from Fig. 3 that there is indeed a discontinuous transition in the large-L limit. We also measured the bulk density, i.e., the density of a smaller hypercube inside the sample whose boundaries are far enough from the surfaces. The bulk density provides a

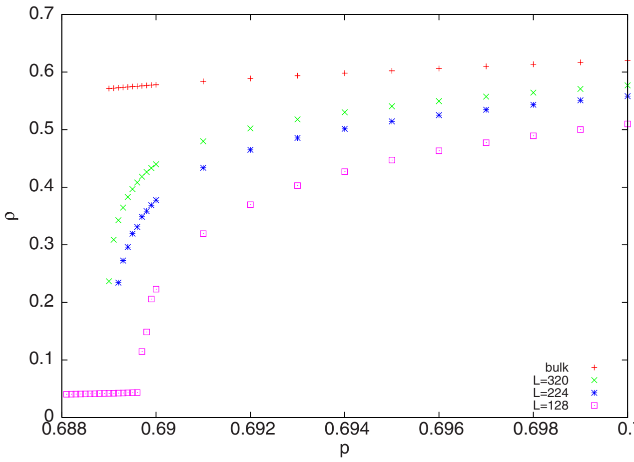

FIG. 3. (Color online) Bulk density and total density vs probability for different sample sizes. The average total density at a given size is a lower bound to the bulk density because of the empty boundary conditions.

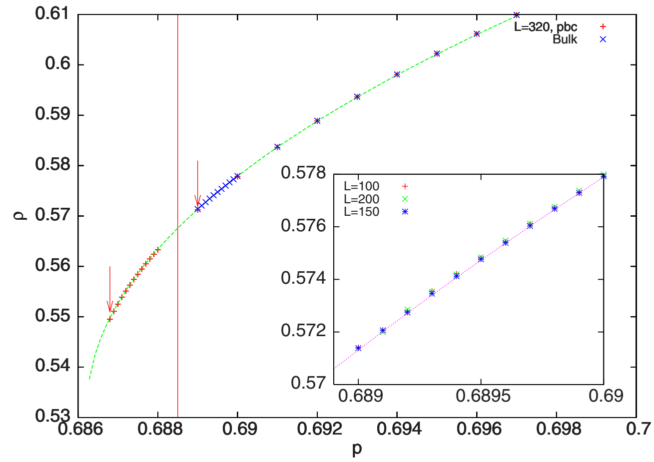

FIG. 4. (Color online) Density vs probability for a sample with periodic boundary conditions and for the bulk of a sample with empty boundary conditions with the same size $L=320$. The data and the fit for periodic boundary conditions are the same as in Fig. 1. The vertical line marks the estimated critical probability $p_c=0.6885(5)$. The two arrows mark the position of the actual transition probabilities. The transition with PBCs is in the unstable region; see text. Inset: magnified view of the bulk density near the transition computed on internal hypercubes of various sizes ($L=100,150,200$) inside a sample with $L=320$.

direct estimate of the density in the thermodynamic limit and strengthens the conclusion that there is a discontinuous transition; see Fig. 3. The transition probability decreases with the sample size and tends to a critical value $p_c=0.6885(5)$ that was estimated through extrapolation. We expect that the interface between the low-density phase on the boundaries and the high-density phase in the bulk penetrates more and more in the sample for $p \to p_c$. As a consequence, in smaller systems the transition will occur at higher values of p.

The percolation value $p_c=0.6885(5)$ is larger than the actual value of the transition in the case of periodic boundary conditions, e.g., $\hat{p}=0.6869$ for $L=320$. In Fig. 4, we plot the bulk density and compare it with the density of the system in the case of periodic boundary conditions. For $p \geqslant p_c$ (marked by a vertical line in the plot), the two densities are equal and we clearly see that the density in the case of periodic boundary conditions is the analytic continuation of the percolating-phase density in the unstable region. We also verified that the correlation length of the bulk is the same as that of the systems with periodic boundary conditions at the corresponding values of p; see Fig. 2. By looking at Figs. 4 and 2, we immediately see that the density and correlation length are regular at the estimated value of the real transition probability $p_c=0.6885$. In particular, using this value of p_c we estimate $\xi_c \approx 2.5$, while in the unstable phase we can observe ξ up to ten lattice spacings.

The qualitative features of the finite-size effect in the case of empty boundaries can be understood in the same way as in thermodynamic first-order transitions, e.g., a ferromagnetic Ising model in a small positive field with the spins on the boundaries forced to be negative. As we noted above, the critical transition probability is shifted to higher values of p, and the true p_c can be estimated through extrapolation (for instance, for $L=320$ the transition is at $p_B=0.689$; see Fig. 4). In general, the finite-size corrections to the density should scale as $1/L$, i.e..,

$$\rho_L(p) = \rho(p) - \frac{1}{L} c_1(p) + O\left(\frac{1}{L^2}\right), \tag{1}$$

and Fig. 3 suggests that the factor $c_1(p)$ diverges at p_c. This factor is determined by the density profiles $\rho(p,z)$ at distance z from an empty surface,

$$c_1(p) = \int_0^\infty [\rho(p) - \rho(p,z)] dz. \tag{2}$$

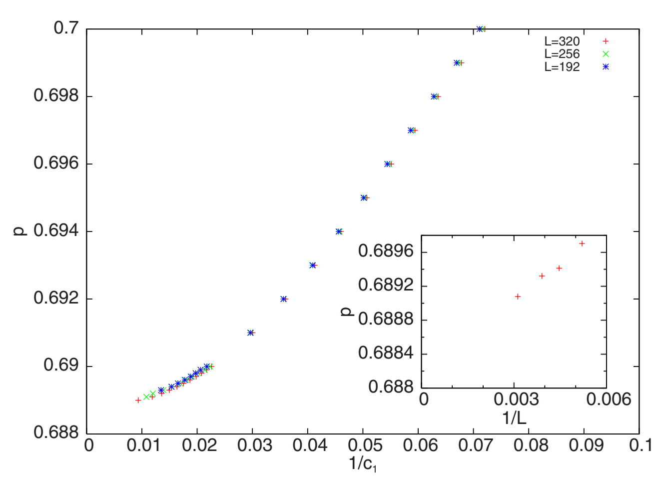

FIG. 5. (Color online) Probability vs inverse of the $1/L$ correction c_1 for different sample sizes. The data are consistent with a divergence of the prefactors at the transition. Inset: probability at which the total density is $\rho=0.3<\rho_c$ as a function of the inverse of the size of the sample; it tends to p_c as the size of the system increases.

Direct inspection of the profiles $\rho(p,z)$ shows that they are consistent with a divergence at the transition, consistently with the expectation that the transition is determined by the penetration depth inside the sample of the interface between the low-density and the percolating phases. The precise nature of the divergence of $c_1(p)$ would require a more detailed analysis, which goes beyond the scope of this work. In Fig. 5, we plot the behavior of the inverse of c_1 and of the transition probability for different sample sizes from which p_c was estimated.

In conclusion, the density of the k-core in four dimensions with $k=4$ exhibits a discontinuous transition at $p_c = 0.6885(5)$ from a high-density percolating phase to a low-density nonpercolating phase. The transition loses its hybrid character with respect to the Bethe lattice case: the density is regular at the transition and the correlation length is finite. These results are most clearly seen numerically considering samples with empty boundary conditions. In finite-size systems with periodic boundary conditions, it is possible to follow the percolating phase in the unstable region $p < p_c = 0.6885(5)$. The behavior of the unstable phase is consistent with an estimated *pseudotransition* at a lower probability $\hat{p}_c \approx 0.6862$. This pseudotransition appears to have a hybrid character with pseudoexponent $\beta = \frac{1}{2}$ and diverging correlation length, but it cannot be observed because even for periodic boundary conditions the unstable phase decays at $p \approx \hat{p} = 0.6869 > \hat{p}_c$ through nucleation of low-density droplets determined by spatial fluctuations of the density.

It would be very interesting to confirm these results through analytical methods. Although the $1/d$ expansion [10] gives no hint about the observed disappearance of the mixed transition, different approaches to perturbation theory around the Bethe solution [21,22] may be able to see it, notwithstanding the possibility that nonperturbative effects should be taken into account.

[1] J. Chalupa, P. L. Leath, and G. R. Reich, J. Phys. C **12**, L31 (1981).
[2] J. Adler, Physica A **171**, 453 (1991).
[3] J. Adler and U. Lev, Braz. J. Phys. **33**, 641 (2003).
[4] R. H. Schonmann, J. Stat. Phys. **58**, 1239 (1990); Physica A **167**, 619 (1990); Ann. Probab. **20**, 174 (1992).
[5] M. Aizenman and J. L. Lebowitz, J. Phys. A **21**, 3801 (1988).
[6] A. Holroyd, Probab. Theory Relat. Fields **125**, 195 (2003).
[7] R. Cerf and F. Manzo, Stochastic Proc. Appl. **101**, 69 (2002); R. Cerf and N. M. Cirillo, Ann. Probab. **27**, 1837 (1999).
[8] A. C. D. van Enter, J. Stat. Phys. **48**, 943 (1987).
[9] P. De Gregorio, A. Lawlor, P. Bradley, and K. A. Dawson, Phys. Rev. Lett. **93**, 025501 (2004).
[10] A. B. Harris and J. M. Schwarz, Phys. Rev. E **72**, 046123 (2005).
[11] P. M. Kogut and P. L. Leath, J. Phys. C **14**, 529 (1981).
[12] J. M. Schwarz, A. J. Liu, and L. Q. Chayes, Europhys. Lett. **73**, 560566 (2006).
[13] M. Sellitto, G. Biroli, and C. Toninelli, Europhys. Lett. **69**, 496 (2005).
[14] C. S. O'Hern, L. E. Silbert, A. J. Liu, and S. R. Nagel, Phys. Rev. E **68**, 011306 (2003); L. E. Silbert, A. J. Liu, and S. R. Nagel, Phys. Rev. Lett. **95**, 098301 (2005).
[15] A. V. Goltsev, S. N. Dorogovtsev, and J. F. F. Mendes, Phys. Rev. E **73**, 056101 (2006).
[16] A. B. Harris and T. C. Lubensky, J. Phys. A **16**, L365 (1983); A. B. Harris, Phys. Rev. B **28**, 2614 (1983).
[17] M. C. Medeiros and C. M. Chaves, Physica A **234**, 604 (1997).
[18] N. S. Branco and C. J. Silva, Int. J. Mod. Phys. C **10**, 921 (1999).
[19] D. Kurtsiefer, Int. J. Mod. Phys. C **14**, 529 (2003).
[20] C. Toninelli, G. Biroli, and D. S. Fisher, Phys. Rev. Lett. **96**, 035702 (2006); C. Toninelli and G. Biroli, e-print arXiv:cond-mat/0512335; M. Jeng and J. M. Schwarz, Phys. Rev. Lett. **98**, 129601 (2007); C. Toninelli, G. Biroli, and D. S. Fisher, *ibid.* **98**, 129602 (2007).
[21] A. Montanari and T. Rizzo, J. Stat. Mech.: Theory Exp. (2005), P10011.
[22] G. Parisi and F. Slanina, J. Stat. Mech.: Theory Exp. (2006), L003.